# ON THE STABILITY OF PERMANENT ROTATIONS OF A QUASI-SYMMETRICAL GYROSTAT

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Sufficient conditions for the stability of permanent rotations of a heavy symmetrical gyrostat were obtained in [1]. Some sufficient conditions are described below.

We consider the motion of a steadily spinning gyrostat with one fixed point $O$. The point $O$ is taken as the origin of a fixed rectangular system of coordinate axes $O\xi\eta\zeta$, with the axis $O\zeta$ directed upwards. A moving rectangular coordinate system $Oxyz$ with origin $O$ is rigidly attached to the solid portion of the gyrostat. The axes $Oxyz$ coincide with the principal axes of inertia, the corresponding moments of inertia about axes $x$, $y$ and $z$ being $A$, $B$ and $C$, respectively.

1. Let the ellipsoid of inertia be an ellipsoid of rotation with $A \neq B = C$. In this case the motion of a heavy gyrostat with one fixed point is described by the system of equations

$$A\frac{dp}{dt} + qc - rb = P(z_0\gamma_2 - y_0\gamma_3), \qquad \frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3 \tag{1.1}$$

$$B\frac{dq}{dt} + (A-B)rp + ra - pc = P(x_0\gamma_3 - z_0\gamma_1), \qquad \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1$$

$$B\frac{dr}{dt} - (A-B)pq + pb - qa = P(y_0\gamma_1 - x_0\gamma_2), \qquad \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2$$

Here $P$ is the weight of the gyrostat; $x_0$, $y_0$ and $z_0$ are the coordinates of its center of gravity $G$; $p$, $q$ and $r$ are the components of the angular velocity $\boldsymbol{\omega}$ on the moving axes; $a$, $b$ and $c$ are the components of the gyrostatic moment $\mathbf{k}$; $\gamma_1$, $\gamma_2$ and $\gamma_3$ are the direction cosines of the axis $O\zeta$ with respect to the moving axes. The equations of motion (1.1) possess the particular solution

$$\gamma_{01} = \alpha = 0, \qquad p_0 = 0, \qquad n = \pm\sqrt{(ay_0 - bx_0)^2 + (cx_0 - az_0)^2} \tag{1.2}$$

$$\gamma_{02} = \beta = \frac{ay_0 - bx_0}{n}, \qquad q_0 = \frac{Px_0}{a}\beta, \qquad \gamma_{03} = \gamma = \frac{az_0 - cx_0}{n}, \qquad r_0 = \frac{Px_0}{a}\gamma$$

describing the permanent rotation of the gyrostat. Here $\alpha$, $\beta$ and $\gamma$ are the direction cosines of the permanent axis of rotation. To each sign of $n$ there corresponds a semi-infinite straight line, which may serve as the permanent axis when it is directed vertically upwards. We take the motion described by (1.2) as the unperturbed state and investigate its stability. Setting

$$p = \xi_1, \quad q = q_0 + \xi_2, \quad r = r_0 + \xi_3, \quad \gamma_1 = \eta_1, \quad \gamma_2 = \beta + \eta_2, \quad \gamma_3 = \gamma + \eta_3$$

in (1.1), we obtain the equations for the perturbed motion, which possess the first integrals

$$V_1 = A\xi_1^2 + B(\xi_2^2 + \xi_3^2 + 2q_0\xi_2 + 2r_0\xi_3) + 2P(x_0\eta_1 + y_0\eta_2 + z_0\eta_3) = \text{const}$$
$$V_2 = A\xi_1\eta_1 + B(\xi_2\eta_2 + \xi_3\eta_3 + q_0\eta_2 + r_0\eta_3 + \beta\xi_2 + \gamma\xi_3) + a\eta_1 + b\eta_2 + c\eta_3 = \text{const}$$
$$V_3 = \eta_1^2 + \eta_2^2 + \eta_3^2 + 2(\beta\eta_2 + \gamma\eta_3) = 0 \qquad (1.3)$$

The Liapunov function $V$ is constructed in the form

$$V = V_1 - \frac{2Px_0}{a}V_2 + \left(B\frac{P^2x_0^2}{a^2} - \frac{Pn}{a}\right)V_3 = A\xi_1^2 + B\xi_2^2 + B\xi_3^2 - 2\frac{Px_0}{a}A\xi_1\eta_1 -$$
$$-2\frac{Px_0}{a}B\xi_2\eta_2 - 2\frac{Px_0}{a}B\xi_3\eta_3 + \left(B\frac{P^2x_0^2}{a^2} - \frac{Pn}{a}\right)(\eta_1^2 + \eta_2^2 + \eta_3^2) \qquad (1.4)$$

According to Sylvester's criterion, the conditions for positive-definiteness of the function (1.4) are the inequalities

$$A > 0, \quad AB > 0, \quad AB^2 > 0, \quad (B - A)\frac{P^2x_0^2}{a^2} - \frac{Pn}{a} > 0$$
$$-\left[(B - A)\frac{P^2x_0^2}{a^2} - \frac{Pn}{a}\right]\frac{Pn}{a} > 0, \qquad \left[(B - A)\frac{P^2x_0^2}{a^2} - \frac{Pn}{a}\right]\frac{P^2n^2}{a^2} > 0 \qquad (1.5)$$

The first three inequalities are always satisfied, while the sixth is a consequence of the fourth and fifth, and the latter two may be written in the form

$$(B - A)Px_0^2 - an > 0, \qquad -an > 0 \qquad (1.6)$$

Since under the conditions (1.6) $V$ is a sign-definite integral of the perturbed motion, according to Liapunov's stability theorem, (1.7) will be the sufficient conditions for the stability of permanent rotations of the gyrostat with respect to the variables $p$, $q$, $r$, $\gamma_1$, $\gamma_2$ and $\gamma_3$. If $A < B = C$, the first of the inequalities (1.6) will be a consequence of the second, and the sufficient condition for stability will be the inequality

$$-an > 0 \qquad (1.7)$$

Since for the given gyrostatic moment the sign of $n$ may always be chosen opposite to the sign of $a$, then the permanent rotation of the gyrostat is stable only for one semi-axis. If, however, the sign of the gyrostatic moment may be arbitrary chosen, then from (1.7) it is clear that stable permanent rotations about both semi-axes may be obtained by choosing the sign of $a$ opposite to the sign of $n$. If $A > B = C$, the second of the inequalities (1.6) follows from the first, and the sufficient condition for stability is

$$-an > (A - B)Px_0^2 \qquad (1.8)$$

If $x_0 = 0$, $y_0 \neq 0$ and $z_0 \neq 0$, then $\omega = 0$, and the sufficient condition for stable equilibrium is the inequality (1.7). Since in this case $n = \pm a\sqrt{y_0^2 + z_0^2}$, then from (1.7) we have

$$\mp a^2\sqrt{y_0^2 + z_0^2} > 0$$

This condition is satisfied only for the lower sign, i.e. the gyrostat is in stable equilibrium if the center of gravity $G$ is below the fixed point $O$ of the gyrostat.

**2.** Let the ellipsoid of inertia be a sphere, i.e. $A = B = C$. In this case the equations of motion (1.1) are obviously very much simplified and, as is well known [2], any straight lines in the plane

$$(bz_0 - cy_0)\alpha + (cx_0 - az_0)\beta + (ay_0 - bx_0)\gamma = 0 \tag{2.1}$$

may serve as the permanent axes of rotation, where $\alpha$, $\beta$ and $\gamma$ are the direction cosines of the semi-axis which may be used for the permanent axis of rotation when it is directed upwards. The angular velocity of rotation is in this case

$$\omega = P\frac{\beta z_0 - \gamma y_0}{\beta c - \gamma b} = P\frac{\gamma x_0 - \alpha z_0}{\gamma\alpha - \alpha c} = P\frac{\alpha y_0 - \beta x_0}{\alpha b - \beta a} \tag{2.2}$$

We consider the particular solution of the system (1.2) for $A = B = C$

$$\alpha = \text{const}, \quad \beta = \text{const}, \quad \gamma = \text{const}, \quad p_0 = x\omega, \quad q_0 = \beta\omega, \quad r_0 = \gamma\omega \tag{2.3}$$

where $\alpha$, $\beta$ and $\gamma$ satisfy (2.1), while $\omega$ is defined by (2.2). We take the motion (2.3) as the unperturbed motion and investigate its stability, assuming that

$$p = p_0 + \xi_1,\ q = q_0 + \xi_2,\ r = r_0 + \xi_3,\ \gamma_1 = \alpha + \eta_1,\ \gamma_2 = \beta + \eta_2,\ \gamma_3 = \gamma + \eta_3$$

in the perturbed motion.

The first integrals of the equations of the perturbed motion are

$$V_1 = A(\xi_1^2 + \xi_2^2 + \xi_3^2) + 2A(p_0\xi_1 + q_0\xi_2 + r_0\xi_3) + 2P(x_0\eta_1 + y_0\eta_2 + z_0\eta_3) = \text{const}$$

$$V_2 = A(\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 + p_0\eta_1 + q_0\eta_2 + r_0\eta_3 + \alpha\xi_1 + \beta\xi_2 + \gamma\xi_3) + a\eta_1 + b\eta_2 + c\eta_3 = \text{const}$$

$$V_3 = \eta_1^2 + \eta_2^2 + \eta_3^2 + 2(\alpha\eta_1 + \beta\eta_2 + \gamma\eta_3) = 0$$

We construct the Liapunov function in the form

$$\begin{aligned} V &= V_1 - 2\omega V_2 + (A\omega^2 + P\lambda)V_3 = \\ &= A(\xi_1^2 + \xi_2^2 + \xi_3^2) - 2A\omega(\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3) + (A\omega^2 + P\lambda)(\eta_1^2 + \eta_2^2 + \eta_3^2) \end{aligned} \tag{2.4}$$

where on the basis of (2.1) and (2.2) we take the constant $\lambda$ to be

$$\lambda = \frac{ay_0 - bx_0}{b\alpha - a\beta} = \frac{bz_0 - cy_0}{c\beta - b\gamma} = \frac{cx_0 - az_0}{a\gamma - c\alpha} \tag{2.5}$$

According to Sylvester's criterion, the condition of positive-definiteness of the function (2.4) is the inequality

$$\lambda > 0 \tag{2.6}$$

When the condition (2.6) is fulfilled, the function (2.4) will be a sign-definite integral of the equations of the perturbed motion, and according to Liapunov's theorem the unperturbed motion will be stable with respect to the parameters $p$, $q$, $r$, $\gamma_1$, $\gamma_2$ and $\gamma_3$.

The sufficient condition (2.6) may be given the following geometrical interpretation: The constant $\lambda$, defined by (2.5), may also be found from Equation

$$\lambda\boldsymbol{\varkappa} \times \mathbf{k} = \mathbf{k} \times \mathbf{OG}$$

where the vector $\mathbf{k}(a, b, c)$ is the gyrostatic moment, $\mathbf{OG}(x_0, y_0, z_0)$ is the radius vector of the center of gravity, and $\boldsymbol{\varkappa}(\alpha, \beta, \gamma)$ is a unit vector along the permanent axis. The inequality (2.6) shows that if the collinear vectors

$$\boldsymbol{\varkappa} \times \mathbf{k},\ \mathbf{k} \times \mathbf{OG} \tag{2.7}$$

have the same direction, the motion is stable. A straight line passing through the fixed point $O$ of the gyrostat parallel to the vector $\mathbf{k}$ divides the plane (2.1) into half-planes. The vectors (2.7) have the same directions if, of the two straight lines forming the axes of permanent rotation, that line which lies in the half-plane not containing the center of gravity is directed upwards.

If

$$\alpha = \frac{x_0}{\pm\sqrt{x_0^2 + y_0^2 + z_0^2}}, \quad \beta = \frac{y_0}{\pm\sqrt{x_0^2 + y_0^2 + z_0^2}}, \quad \gamma = \frac{z_0}{\pm\sqrt{x_0^2 + y_0^2 + z^2}}$$

we have $\omega = 0$, and the gyrostat is in equilibrium. In that case

$$\lambda = \mp\sqrt{x_0^2 + y_0^2 + z_0^2}$$

and the sufficient condition (2.6) for stable equilibrium is satisfied if the center of gravity is vertically below the point of support of the gyrostat.

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